Geometric discretization of the Koenigs nets

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Abstract

We introduce the Koenigs lattice, which is a new integrable reduction of the quadrilateral lattice (discrete conjugate net) and provides natural integrable discrete analogue of the Koenigs net. We construct the Darboux-type transformation of the Koenigs lattice and we show permutability of superpositions of such transformations, thus proving integrability of the Koenigs lattice. We also investigate the geometry of the discrete Koenigs transformation. In particular we characterize the Koenigs transformation in terms of an involution determined by a congruence conjugate to the lattice.

Keywords: discrete geometry; integrable systems; quadrilateral lattices; Darboux transformations; Koenigs nets

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1 Introduction

In the XIX-th century one of the most favorite subjects of the differential geometry [1, 4] was investigation of special classes of surfaces (or, more appropriate, coordinate systems on surfaces and submanifolds) which allow for transformations exhibiting the so called *permutability property.* Such transformations called, depending on the context, the Darboux, Bianchi, Bäcklund, Laplace, Moutard, Koenigs, Combescure, Lévy, Ribaucour or the fundamental transformation of Jonas, can be also described in terms of certain families of lines called line congruences [16, 17]. It turns out that most of the "interesting" submanifolds is provided by reductions of conjugate nets, and the transformations between such submanifolds are the corresponding reductions of the fundamental (or Jonas) transformation of the conjugate nets.

From the other side such submanifolds are described by solutions of certain nonlinear partial differential equations, which turn out to be extensively studied in the modern theory of integrable systems; here also the existence of transformations (called in this context the Darboux transformations) appears to be essential. For example, the conjugate nets, their iso-conjugate deformations and transformations are described [10] by the so called multicomponent Kadomtsev–Petviashvilii hierarchy, which is considered often as the basic system of equations of the soliton theory [5, 19].

Recently the integrable discrete (difference) versions of integrable differential equations attracted a lot of attention (see, for example, articles in [25, 3, 24, 2]). The interest in discrete integrable systems is stimulated from various directions, like numerical methods, theory of special functions, but also from statistical and quantum physics [18, 22]. The discrete integrable systems are considered more fundamental then the corresponding differential systems. Discrete equations include the continuous theory as the result of a limiting procedure, moreover different limits may give from one

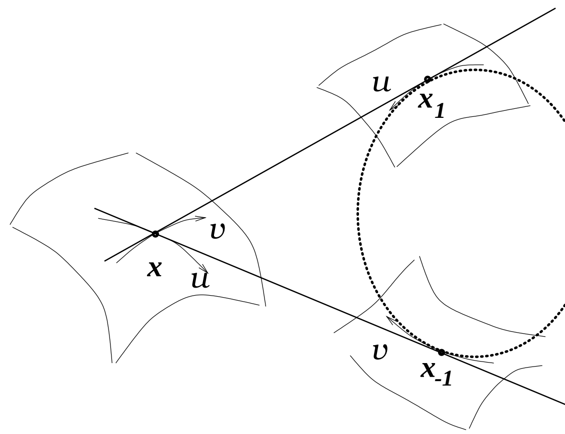

Figure 1: The Koenigs net

discrete equation various differential ones. Furthermore, discrete equations reveal some symmetries lost in the continuous limit.

During last few years the connection between geometry and integrability has been observed also at a discrete level. The present paper is the next one in the series of attempts to construct the integrable discrete geometry — the theory of lattice submanifolds described by integrable difference equations.

The natural discrete analogs of certain coordinate systems on surfaces were studied by Sauer [32]. In particular, he introduced the discrete conjugate nets in $\mathbb{E}^3$ as lattices with planar elementary quadrilaterals. The importance of the discrete conjugate nets (the quadrilateral lattices) in the soliton theory was recognized recently in [6, 12]. The Darboux-type transformations of the quadrilateral lattices have been found in [26], and the geometry of these transformations was investigated in detail in [14]. In the literature [9, 21, 7, 13, 8] there are known various integrable reductions of the quadrilateral lattices. We introduce here the discrete analogue of the Koenigs reduction of conjugate nets. Let us state briefly the main definitions, ideas and results of this paper.

Consider generic two-dimensional conjugate net [23] in M-dimensional projective space $\mathbb{P}^M$. The homogeneous coordinates $\boldsymbol{x}(u,v) \in \mathbb{R}^{M+1}_*$ of the net satisfy the Laplace equation

$$\boldsymbol{x}_{,uv} = a\boldsymbol{x}_{,u} + b\boldsymbol{x}_{,v} + c\boldsymbol{x}, \tag{1.1}$$

where comma denotes differentiation (e.g., $\boldsymbol{x}_{,u} = \frac{\partial \boldsymbol{x}}{\partial u}$), and a, b, c are functions of the conjugate parameters (u,v) of the net. Its Laplace transforms

$$\boldsymbol{x}_1 = \boldsymbol{x}_{,v} - a\boldsymbol{x}, \qquad \boldsymbol{x}_{-1} = \boldsymbol{x}_{,u} - b\boldsymbol{x}, \tag{1.2}$$

are another conjugate nets such that the v-tangents of $\boldsymbol{x}$ coincide with the corresponding u-tangent lines of $\boldsymbol{x}_1$ and that the u-tangents of $\boldsymbol{x}$ coincide with the corresponding v-tangent lines of $\boldsymbol{x}_{-1}$ (see figure 1). In the tangent plane at a point $\boldsymbol{x}$ there is a linear system (pencil) of conics tangent to the u-coordinate line at the point $\boldsymbol{x}_1$ and tangent to the v-curve at the point $\boldsymbol{x}_{-1}$. When there is one conic of this pencil with the second order contact with the u-curve of $\boldsymbol{x}_1$ and with the second order contact with the v-curve of $\boldsymbol{x}_{-1}$ then the net is called *the net of Koenigs.* It turns out that the Laplace equation (1.1) of the Koenigs net can be gauged into the form

$$\boldsymbol{x}_{,uv} = f\boldsymbol{x}. \tag{1.3}$$

The integrable discrete analogue of two-dimensional conjugate net is a $\mathbb{Z}^2$-lattice made of planar quadrilaterals [32, 6]. One can construct for such lattices [6] the analogue of the Laplace transforms

$\boldsymbol{x}_1$ and $\boldsymbol{x}_{-1}$. In the plane of the elementary quadrilateral at a point $\boldsymbol{x}(n_1, n_2)$ there is a pencil of conics passing through the points $\boldsymbol{x}_1(n_1, n_2)$ and $\boldsymbol{x}_1(n_1+1, n_2)$ of the Laplace transform $\boldsymbol{x}_1$ and passing through the points $\boldsymbol{x}_{-1}(n_1, n_2)$ and $\boldsymbol{x}_{-1}(n_1, n_2+1)$ of the Laplace transform $\boldsymbol{x}_{-1}$; we have replaced the tangency to a curve by its natural discrete analogue of passing through two neighbouring points of the discrete parametric curve. When there is one conic of this pencil passing through $\boldsymbol{x}_1(n_1+2, n_2)$ and passing through $\boldsymbol{x}_{-1}(n_1, n_2+2)$ then we call such a lattice *the Koenigs lattice*.

The reduction of the fundamental transformation of conjugate nets to the class of the Koenigs nets is called *the transformation of Koenigs*. Such transformation is determined only by the half of the data of the fundamental transformation: given congruence conjugate to the Koenigs net then the second Koenigs net $\boldsymbol{x}'$ is the harmonic conjugate of $\boldsymbol{x}$ with respect to the focal nets of the congruence. The discrete analogue of this construction is more subtle. One can show that $\boldsymbol{x}'$ is the image of $\boldsymbol{x}$ in an involution on the corresponding line of the congruence. This involution is uniquely defined by the focal nets of the congruence; in the continuous case the focal points are the double points of the involution [16, 23]

We have sketched the main ideas and results of the paper. The detailed presentation will be as follows. In Section 2 we define and study in detail the Koenigs lattice. In particular we discuss the integrability of the Koenigs lattice from the point of view of the Pascal theorem. In Section 3 we present the discrete analogue of the Koenigs transformation and then in Section 4 we investigate geometric properties of the Koenigs transformation. Finally, in Section 5 we investigate superpositions of the Koenigs transformation and prove their permutability, thus showing integrability of the Koenigs lattice.

2 The Koenigs lattice

Consider a two-dimensional quadrilateral lattice in M-dimensional projective space $\mathbb{P}^M$, whose points labelled by two-dimensional integer lattice $\mathbb{Z}^2$, satisfy the property of planarity of elementary quadrilaterals [32, 6]. In terms of the homogeneous coordinates such a lattice is described by solution of the discrete Laplace equation

$$\boldsymbol{x}_{(12)} = A_{(1)}\boldsymbol{x}_{(1)} + B_{(2)}\boldsymbol{x}_{(2)} + C\boldsymbol{x}, \tag{2.1}$$

where $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{R}^{M+1}_*$ and subscripts in brackets mean shifts along the $\mathbb{Z}^2$ lattice, i.e., $\boldsymbol{x}_{(\pm 1)}(n_1, n_2) = \boldsymbol{x}(n_1 \pm 1, n_2)$, $\boldsymbol{x}_{(\pm 2)}(n_1, n_2) = \boldsymbol{x}(n_1, n_2 \pm 1)$, and $\boldsymbol{x}_{(\pm 1 \pm 2)}(n_1, n_2) = \boldsymbol{x}(n_1 \pm 1, n_2 \pm 1)$. Here also A, B and C are functions on $\mathbb{Z}^2$ which characterize the lattice completely up to initial curves $\boldsymbol{x}(n_1, 0)$ and $\boldsymbol{x}(0, n_2)$. Notice that multiplication of $\boldsymbol{x}$ by a non-zero function ρ implies the corresponding change of A, B and C but does not change the lattice itself.

As it was shown in [6] because of the planarity of the elementary quadrilaterals of the lattice one can define its Laplace transforms $\boldsymbol{x}_1$ and $\boldsymbol{x}_{-1}$ (see figure 2)

$$\boldsymbol{x}_1 = \boldsymbol{x}_{(2)} - A\boldsymbol{x}, \qquad \boldsymbol{x}_{-1} = \boldsymbol{x}_{(1)} - B\boldsymbol{x}. \tag{2.2}$$

Remark. Notice that our notation differs from that of [6] by opposite ordering of the transformations and by shifts in parameters. Moreover in [6] we used the affine gauge which made some formulas more complicated.

Using the discrete Laplace equation (2.1) one can show that $\boldsymbol{x}_{1(1)}$ is collinear with $\boldsymbol{x}$, $\boldsymbol{x}_{(2)}$ and $\boldsymbol{x}_1$

$$\boldsymbol{x}_{1(1)} = B_{(2)}\boldsymbol{x}_{(2)} + C\boldsymbol{x} = B_{(2)}\boldsymbol{x}_1 + (B_{(2)}A + C)\boldsymbol{x}; \tag{2.3}$$

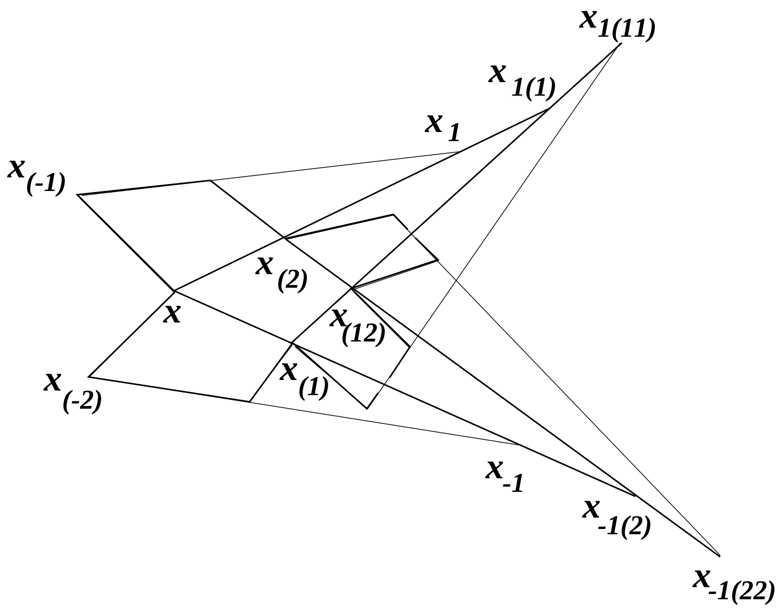

Figure 2: The quadrilateral lattice and its Laplace transforms

similarly $\boldsymbol{x}_{-1(2)}$ is collinear with $\boldsymbol{x}$, $\boldsymbol{x}_{(1)}$ and $\boldsymbol{x}_{-1}$

$$\boldsymbol{x}_{-1(2)} = A_{(1)}\boldsymbol{x}_{(1)} + C\boldsymbol{x} = A_{(1)}\boldsymbol{x}_{-1} + (A_{(1)}B + C)\boldsymbol{x}. \tag{2.4}$$

Remark. Two functions H and K defined [6] as the cross-ratios

$$H \quad = cr(\boldsymbol{x}_{(1)}, \boldsymbol{x}; \boldsymbol{x}_{-1}, \boldsymbol{x}_{-1(2)}) = -\tfrac{A_{(1)}B}{C}, \tag{2.5}$$

$$K \quad = cr(\boldsymbol{x}_{(2)}, \boldsymbol{x}; \boldsymbol{x}_{1}, \boldsymbol{x}_{1(1)}) = -\tfrac{B_{(2)}A}{C}, \tag{2.6}$$

are gauge-invariant and are called the invariants of the lattice $\boldsymbol{x}$. They are natural discrete analogues of the invariants

$$h = c + ab - a_{,u}, \qquad k = c + ab - b_{,v}, \tag{2.7}$$

of conjugate nets. In the continuous case the Koenigs nets have equal invariants, i.e., $h = k$. This property does not transfer to the discrete case.

It is well known (see for example [31]) that five distinct points in a projective plane, no four of which are collinear, uniquely determine a conic. Moreover, a pencil of conics (one-dimensional linear subspace of the five dimensional space of conics) is uniquely determined by four points (the base of the pencil) no three of which are collinear.

The four points $\boldsymbol{x}_1$, $\boldsymbol{x}_{1(1)}$, $\boldsymbol{x}_{-1}$ and $\boldsymbol{x}_{-1(2)}$ belong to the plane $P_{\boldsymbol{x}\boldsymbol{x}_{(1)}\boldsymbol{x}_{(2)}}$ of the elementary quadrilateral of $\boldsymbol{x}$ and define a linear system of conics. Let us choose the points $\boldsymbol{x}$, $\boldsymbol{x}_{-1(2)}$, $\boldsymbol{x}_{1(1)}$ as vertices of the local triangle of reference in that plane, i.e., a point $y_1\boldsymbol{x} + y_2\boldsymbol{x}_{-1(2)} + y_3\boldsymbol{x}_{1(1)}$ has coordinates proportional to (y_1, y_2, y_3). Then the equation of a general conic of the pencil is of the form

$$y_1^2 + (A_{(1)}B + C)y_1y_2 + (B_{(2)}A + C)y_1y_3 + \lambda y_2y_3 = 0, \tag{2.8}$$

with λ being a parameter.

Definition 2.1. *The Koenigs lattice* is a two dimensional quadrilateral lattice such that for every point $\boldsymbol{x}$ of the lattice there exist a conic passing through the six points $\boldsymbol{x}_1$, $\boldsymbol{x}_{1(1)}$,$\boldsymbol{x}_{1(11)}$, $\boldsymbol{x}_{-1}$, $\boldsymbol{x}_{-1(2)}$ and $\boldsymbol{x}_{-1(22)}$.

Proposition 2.1. *The Laplace equation of the Koenigs lattice can be gauged into the canonical form*

$$\boldsymbol{x}_{(12)} + \boldsymbol{x} = F_{(1)}\boldsymbol{x}_{(1)} + F_{(2)}\boldsymbol{x}_{(2)}. \tag{2.9}$$

Proof. The points $\boldsymbol{x}_{1(11)}$ and $\boldsymbol{x}_{-1(22)}$ also belong to the plane $P_{\boldsymbol{x}\boldsymbol{x}_{(1)}\boldsymbol{x}_{(2)}} = P_{\boldsymbol{x}\boldsymbol{x}_{-1(2)}\boldsymbol{x}_{1(1)}}$ and have the following decompositions

$$\boldsymbol{x}_{1(11)} = -\left(CB_{(12)} + \frac{CC_{(1)}}{A_{(1)}}\right)\boldsymbol{x} + \left(B_{(12)} + \frac{C_{(1)}}{A_{(1)}}\right)\boldsymbol{x}_{-1(2)} + B_{(12)}\boldsymbol{x}_{1(1)},$$
$$\boldsymbol{x}_{-1(22)} = -\left(CA_{(12)} + \frac{CC_{(2)}}{B_{(2)}}\right)\boldsymbol{x} + A_{(12)}\boldsymbol{x}_{-1(2)} + \left(A_{(12)} + \frac{C_{(2)}}{B_{(2)}}\right)\boldsymbol{x}_{1(1)}.$$

In the pencil (2.8) there exist a conic passing through $\boldsymbol{x}_{1(11)}$ and $\boldsymbol{x}_{-1(22)}$ if and only if the coefficients of the Laplace equation (2.1) of the Koenigs lattice satisfy the constraint

$$\frac{AC_{(2)}}{A_{(12)}} = \frac{BC_{(1)}}{B_{(12)}}. \tag{2.10}$$

This constraint is the compatibility condition of the linear system for the unknown function ρ

$$\rho_{(12)} = -C\rho, \tag{2.11}$$
$$\rho_{(1)}A = \rho_{(2)}B. \tag{2.12}$$

Using solution of this system as the gauge function we obtain new representation

$$\tilde{\boldsymbol{x}} = \frac{1}{\rho}\boldsymbol{x}, \tag{2.13}$$

of the Koenigs lattice which satisfies equation (2.9) with

$$F = \frac{A\rho}{\rho_{(2)}} = \frac{B\rho}{\rho_{(1)}}. \tag{2.14}$$

□

Remark. Equation (2.9), in a gauge equivalent form, appeared first in [29, 11] in connection with the integrable discretization of the Bianchi–Ernst system.

Usually the integrability of a nonlinear problem is connected with its hidden linear structure. It turns out that, with the help of the celebrated Pascal theorem on six points on a conic, the discrete Koenigs constraint can be formulated in a linear way.

Proposition 2.2. *The quadrilateral lattice $\boldsymbol{x}$ is the Koenigs lattice if and only if the lines $L_{\boldsymbol{x}_{-1}\boldsymbol{x}_{1(11)}}$, $L_{\boldsymbol{x}_1\boldsymbol{x}_{-1(22)}}$ and $L_{\boldsymbol{x}\boldsymbol{x}_{(12)}}$ intersect in a single point.*

Proof. Recall that given six point 1, 2, 3, 4, 5, 6 belong to a conic if and only if the points $i = L_{12} \cap L_{45}$, $j = L_{23} \cap L_{56}$ and $k = L_{34} \cap L_{61}$ are collinear. We apply the Pascal theorem to the six points $\boldsymbol{x}_{-1}$, $\boldsymbol{x}_{-1(2)}$, $\boldsymbol{x}_{-1(22)}$, $\boldsymbol{x}_1$, $\boldsymbol{x}_{1(1)}$ and $\boldsymbol{x}_{1(11)}$. Because point $\boldsymbol{x}$ is the intersection of lines $L_{\boldsymbol{x}_{-1}\boldsymbol{x}_{-1(2)}}$ and $L_{\boldsymbol{x}_1\boldsymbol{x}_{1(1)}}$ and the point $\boldsymbol{x}_{(12)}$ is the intersection of lines $L_{\boldsymbol{x}_{-1(2)}\boldsymbol{x}_{-1(22)}}$ and $L_{\boldsymbol{x}_{1(1)}\boldsymbol{x}_{1(11)}}$ then there exists a conic passing through the six points if and only if the statement of the proposition holds. □

3 The discrete Koenigs transformation

The Koenigs transformation is the reduction of the fundamental transformation to the class of the Koenigs nets and lattices. Let us first recall relevant definitions and results from the theory of transformations of quadrilateral lattices [14]. Then we present the algebraic definition of the Koenigs reduction of the fundamental transformation. We postpone to next section the discussion of the geometric interpretation of the Koenigs transformation.

3.1 The fundamental transformation of quadrilateral lattices

We recall the basic results from the theory of transformations of quadrilateral lattices [14]. The novelty here is the description of the theory in the homogeneous formalism (but the geometric content does not change). We constraint our presentation to two dimensional lattices and congruences only.

Definition 3.1. The discrete two-dimensional line congruence L is a $\mathbb{Z}^2$-family of lines in $\mathbb{P}^M$ such that any two neighbouring lines intersect. The intersection $\boldsymbol{y}_i = L \cap L_{(-i)}$, $i = 1, 2$, is called the i-th focal lattice of the congruence.

Corollary 3.1. *The focal lattices of discrete two-dimensional line congruences are quadrilateral lattices.*

Definition 3.2. A two-dimensional quadrilateral lattice $\boldsymbol{x}$ and a two-dimensional congruence L are called conjugate when points of the lattice belong to the corresponding lines of the congruence, i.e., $\boldsymbol{x}(n_1, n_2) \in L(n_1, n_2)$ for all $(n_1, n_2) \in \mathbb{Z}^2$.

Definition 3.3. The quadrilateral lattice $\boldsymbol{x}'$ is a fundamental transform of $\boldsymbol{x}$ if there exists a congruence (called the congruence of the transformation) conjugate to both lattices.

Theorem 3.2. *Two quadrilateral lattices $\boldsymbol{x}$ and $\boldsymbol{x}'$ are fundamental transforms of each other if and only if there exist solutions ϕ and ϕ' of the Laplace equations of the lattices and there exist functions k and ℓ such that the system*

$$\Delta_1 \left(\frac{\boldsymbol{x}'}{\phi'} \right) = k_{(1)} \Delta_1 \left(\frac{\boldsymbol{x}}{\phi} \right), \tag{3.1}$$

$$\Delta_2 \left(\frac{\boldsymbol{x}'}{\phi'} \right) = \ell_{(2)} \Delta_2 \left(\frac{\boldsymbol{x}}{\phi} \right), \tag{3.2}$$

is satisfied.

Corollary 3.3. *The system* (3.1) *is compatible if and only if there exist a solution θ of the equation*

$$C\theta_{(12)} = -B\theta_{(1)} - A\theta_{(2)} + \theta, \tag{3.3}$$

called the adjoint of (2.1)*, and the functions k and ℓ are solution of the following system*

$$\begin{aligned} k - \ell &= \phi\theta, \\ \Delta_1 \ell &= -(\phi_{(1)} - B\phi)\theta_{(1)}, \\ \Delta_2 k &= (\phi_{(2)} - A\phi)\theta_{(2)}. \end{aligned} \tag{3.4}$$

Corollary 3.4. *The fundamental transformation of the given lattice $\boldsymbol{x}$ can be constructed when we are given a solution of its Laplace equation and a solution of its adjoint (both are given up to two functions of single variables). The next step is to find the functions k and ℓ (given up to a constant) by solving the system* (3.4)*. Finally, the transformed lattice in the gauge*

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}'}{\phi'}, \tag{3.5}$$

is obtained (up to a constant vector) from the system (3.1)*. The coefficients of the Laplace equation of the lattice $\hat{\boldsymbol{x}}$ read*

$$\hat{A} = A\frac{k_{(2)}\phi}{k\phi_{(2)}}, \qquad \hat{B} = B\frac{\ell_{(1)}\phi}{\ell\phi_{(1)}}, \qquad \hat{C} = 1 - \hat{A}_{(1)} - \hat{B}_{(2)}. \tag{3.6}$$

Remark. The corresponding tangent lines of $\boldsymbol{x}$ and $\boldsymbol{x}'$ intersect in points of the quadrilateral lattices

$$\mathcal{L}_1(\boldsymbol{x}) = \Delta_1\left(\frac{\boldsymbol{x}}{\phi}\right), \qquad \mathcal{L}_2(\boldsymbol{x}) = \Delta_2\left(\frac{\boldsymbol{x}}{\phi}\right), \tag{3.7}$$

called the Lévy transforms [14] of $\boldsymbol{x}$.

Corollary 3.5. *The focal lattices of the congruence of the transformation given by*

$$\boldsymbol{y}_1 = k\frac{\boldsymbol{x}}{\phi} - \frac{\boldsymbol{x}'}{\phi'}, \qquad \boldsymbol{y}_2 = \ell\frac{\boldsymbol{x}}{\phi} - \frac{\boldsymbol{x}'}{\phi'}, \tag{3.8}$$

satisfy equations

$$\begin{aligned} \boldsymbol{y}_1 - \boldsymbol{y}_2 &= \theta\boldsymbol{x}, \\ \Delta_1\boldsymbol{y}_2 &= -(\boldsymbol{x}_{(1)} - B\boldsymbol{x})\theta_{(1)}, \\ \Delta_2\boldsymbol{y}_1 &= (\boldsymbol{x}_{(2)} - A\boldsymbol{x})\theta_{(2)}, \end{aligned} \tag{3.9}$$

i.e., they can be found using the solution θ of the adjoint equation (3.3) *only.*

Remark. Equations (3.9) can be used to find congruences conjugate to the lattice $\boldsymbol{x}$. Notice that the role of the new solution ϕ of the Laplace equation (2.1) of the lattice $\boldsymbol{x}$ in equations (3.4) is taken in (3.9) by $\boldsymbol{x}$ itself.

Remark. The lattices $\boldsymbol{y}_1 = \mathcal{L}_1^*(\boldsymbol{x})$ and $\boldsymbol{y}_2 = \mathcal{L}_2^*(\boldsymbol{x})$ are also called the adjoint Lévy transforms [14] of $\boldsymbol{x}$.

3.2 The algebraic formulation of the discrete Koenigs transformation

Proposition 3.6. *Given the Koenigs lattice $\boldsymbol{x}$ satisfying equation* (2.9) *and given a scalar solution θ of its adjoint equation (the Moutard equation)*

$$\theta_{(12)} + \theta = F(\theta_{(1)} + \theta_{(2)}), \tag{3.10}$$

then the solution $\boldsymbol{x}'$ of the linear system

$$\Delta_1\left(\frac{\boldsymbol{x}'}{\phi'}\right) = (\theta\theta_{(2)})_{(1)}\Delta_1\left(\frac{\boldsymbol{x}}{\phi}\right), \tag{3.11}$$

$$\Delta_2\left(\frac{\boldsymbol{x}'}{\phi'}\right) = -(\theta\theta_{(1)})_{(2)}\Delta_2\left(\frac{\boldsymbol{x}}{\phi}\right), \tag{3.12}$$

with

$$\phi = \theta_{(1)} + \theta_{(2)}, \qquad \phi' = \frac{1}{\theta_{(1)}} + \frac{1}{\theta_{(2)}}, \tag{3.13}$$

is a new Koenigs lattice satisfying equation (2.9) *with*

$$F' = F\frac{\theta_{(1)}\theta_{(2)}}{\theta\theta_{(12)}}. \tag{3.14}$$

Proof. First one should observe [29] that if θ satisfies the Moutard equation (3.10) then $\phi = \theta_{(1)}+\theta_{(2)}$ is a solution of the Koenigs lattice equation (2.9). Then the corresponding solutions of the system (3.4) are

$$k = \theta_{(2)}\theta, \qquad \ell = -\theta_{(1)}\theta, \tag{3.15}$$

and the coefficients of the Laplace equation of the new quadrilateral lattice $\hat{\boldsymbol{x}}$ read

$$\begin{aligned} \hat{A} &= F\frac{\theta_{(1)}+\theta_{(2)}}{\theta\theta_{(2)}}\left(\frac{\theta\theta_{(2)}}{\theta_{(1)}+\theta_{(2)}}\right)_{(2)}, \\ \hat{B} &= F\frac{\theta_{(1)}+\theta_{(2)}}{\theta\theta_{(1)}}\left(\frac{\theta\theta_{(1)}}{\theta_{(1)}+\theta_{(2)}}\right)_{(1)}, \\ \hat{C} &= 1-\hat{A}_{(1)}-\hat{B}_{(2)}. \end{aligned} \tag{3.16}$$

The function

$$\rho = \frac{\theta_{(1)}\theta_{(2)}}{\theta_{(1)}+\theta_{(2)}} = \frac{1}{\phi'}, \tag{3.17}$$

is a solution of the system (2.11)-(2.12). This implies that

$$\boldsymbol{x}' = \phi'\hat{\boldsymbol{x}}, \tag{3.18}$$

satisfies the Koenigs lattice equation (2.9), and the corresponding potential, according to equation (2.14), agrees with that given by (3.14). □

Corollary 3.7. *The function ϕ' satisfies the Laplace equation of the lattice $\boldsymbol{x}'$.*

Remark. The important observation that the adjoint of the Koenigs lattice equation is the Moutard equation is due to Nieszporski [27].

4 The geometric meaning of the Koenigs transformation

To present the geometric meaning of the discrete Koenigs transformation, introduced in the previous section, we first recall standard results on involutions on a projective line L (see [31]).

Theorem 4.1. *If a projective transformation $h : L \to L$ has two distinct fixed points $\boldsymbol{p}$ and $\boldsymbol{q}$ then h is an involution if and only if for any point $\boldsymbol{u} \in L$ its image $h(\boldsymbol{u})$ is the harmonic conjugate of $\boldsymbol{u}$ with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$.*

Theorem 4.2. *A projective involution is uniquely determined giving two pairs of homologous points.*

4.1 The discrete Koenigs transformation as geometric reduction of the fundamental transformation

Proposition 4.3. *Given Koenigs lattice $\boldsymbol{x}$ and its transform $\boldsymbol{x}'$, denote by $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ the focal lattices of the congruence of the transformation. The Koenigs transform $\boldsymbol{x}'$ is the image of $\boldsymbol{x}$ in the unique involution mapping $\boldsymbol{y}_1$ into $\boldsymbol{y}_{1(1)}$ and $\boldsymbol{y}_2$ into $\boldsymbol{y}_{2(2)}$.*

Proof. In the gauge of Proposition 3.6 and due to Corollary 3.5 we have

$$\boldsymbol{x} = \frac{1}{\theta}\left(\boldsymbol{y}_1 - \boldsymbol{y}_2\right), \qquad \boldsymbol{x}' = -\frac{1}{\theta_{(1)}\theta_{(2)}}\left(\theta_{(1)}\boldsymbol{y}_1 + \theta_{(2)}\boldsymbol{y}_2\right). \tag{4.1}$$

Equations (3.9) imply that

$$\boldsymbol{y}_{1(1)} = \frac{1}{\theta}\left(F\theta_1\boldsymbol{y}_1 - (F\theta_1 - \theta)\boldsymbol{y}_2\right), \tag{4.2}$$

$$\boldsymbol{y}_{2(2)} = \frac{1}{\theta}\left(F\theta_2\boldsymbol{y}_2 - (F\theta_2 - \theta)\boldsymbol{y}_1\right). \tag{4.3}$$

The unique involution mapping $\boldsymbol{y}_1$ into $\boldsymbol{y}_{1(1)}$ and $\boldsymbol{y}_2$ into $\boldsymbol{y}_{2(2)}$ is the projection of the linear map, which is convenient to choose in the form

$$\boldsymbol{y}_1 \mapsto -F\boldsymbol{y}_1 + \left(F - \frac{\theta}{\theta_1}\right)\boldsymbol{y}_2, \tag{4.4}$$

$$\boldsymbol{y}_2 \mapsto F\boldsymbol{y}_2 - \left(F - \frac{\theta}{\theta_2}\right)\boldsymbol{y}_1. \tag{4.5}$$

As one can check directly the image of $\boldsymbol{x}$ in this mapping is $\boldsymbol{x}'$. □

Corollary 4.4. *In the continuous limit the points of the focal lattices become the double points of the involution. This fact and Theorem 4.1 imply that $\boldsymbol{x}'$ becomes, in the limit, the harmonic conjugate of $\boldsymbol{x}$ with respect to the pair $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$.*

It turns out that the property of the Koenigs transformation described in Proposition 4.3 holds exclusively for the Koenigs lattice and selects it from general quadrilateral lattices.

Proposition 4.5. *Given two quadrilateral lattices $\boldsymbol{x}$ and $\boldsymbol{x}'$ related by the fundamental transformation such that the focal lattices $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ of the congruence of the transformation do not degenerate to a single point. If $\boldsymbol{x}'$ is the image of $\boldsymbol{x}$ in the unique involution mapping $\boldsymbol{y}_1$ into $\boldsymbol{y}_{1(1)}$ and $\boldsymbol{y}_2$ into $\boldsymbol{y}_{2(2)}$ then $\boldsymbol{x}$ and $\boldsymbol{x}'$ are Koenigs lattices related by the Koenigs transformation.*

Proof. Notice first that the excluded situation of the degenerated focal lattices corresponds to the trivial solution $\theta = 0$ of the adjoint equation (3.3) of the lattice $\boldsymbol{x}$.

Formulas (3.8) and the assumption of the proposition imply the following constraint on the functions k and ℓ

$$k_{(1)}k = \ell_{(2)}\ell. \tag{4.6}$$

This constraint allows to solve the system (3.4)

$$k = \frac{\phi\theta\theta_{(2)}A}{A\theta_{(2)} + B\theta_{(1)}}, \qquad \ell = -\frac{\phi\theta\theta_{(1)}B}{A\theta_{(2)} + B\theta_{(1)}}, \tag{4.7}$$

and gives the following relation between the coefficients A and B of the Laplace equation (2.1)

$$kB\theta_{(1)} + \ell A\theta_{(2)} = 0. \tag{4.8}$$

In consequence we have also

$$k_{(1)} = -\frac{\phi\theta_{(1)}\theta_{(12)}BC}{A\theta_{(2)} + B\theta_{(1)}}, \qquad \ell_{(2)} = \frac{\phi\theta_{(2)}\theta_{(12)}AC}{A\theta_{(2)} + B\theta_{(1)}}. \tag{4.9}$$

The above relations allow to check that the condition (2.10) holds for the lattice $\boldsymbol{x}$, which implies that $\boldsymbol{x}$ is the Koenigs lattice (notice that due to the symmetry between both lattices in the definition of the fundamental transformation and in the notion of the harmonic conjugate the analogous condition holds for the lattice $\boldsymbol{x}'$).

Assuming therefore that the function $\boldsymbol{x}$ of the first lattice is in the canonical gauge $A = B = F$, $C = -1$, one obtains from equations (4.7)-(4.9) that

$$k = \theta\theta_{(2)}N_2(n_2), \qquad \ell = \theta\theta_{(1)}N_1(n_1), \tag{4.10}$$

where $N_i(n_i)$, $i = 1, 2$, are functions (still to be determined) of single variables. Then the function

$$\phi = \theta_{(1)}N_1(n_1) - \theta_{(2)}N_2(n_2), \tag{4.11}$$

obtained using (3.4), satisfies equation (2.9) for generic F only if $N_1 = -N_2 = const$. Without loss of generality this constant can be put equal to 1. The rest of the proof is the same like the proof of Proposition 3.6. □

Remark. The above proposition in the continuous case was found by Koenigs [20].

Remark. The excluded case $\theta = 0$ corresponds to the reduction of the fundamental transformation to the radial transformation [14].

4.2 Further geometric properties of the discrete Koenigs transformation

To understand more the relation between the Koenigs lattice, as defined in terms of conics, and the geometric description of the Koenigs transformation in terms of involutions on lines of the congruence, we will need the following result.

Theorem 4.6 (Desargues—Sturm). *A pencil of conics of a projective plane determines a projective involution on every line that does not intersect the base of the pencil.*

Remark. The results presented in this section are generalization to the discrete level of the results of Tzitzéica [34] and Eisenhart [15].

It turns out that the Koenigs transformation defines certain family of quadrics. This family contains the pencils of conics of the both Koenigs lattices $\boldsymbol{x}$ and $\boldsymbol{x}'$.

Proposition 4.7. *Given Koenigs lattice $\boldsymbol{x}$ and its transform $\boldsymbol{x}'$ then the pencils of conics of both lattices determine the same involution on the intersection line of the planes of elementary quadrilaterals of $\boldsymbol{x}$ and $\boldsymbol{x}'$.*

Proof. In the gauge such that $\boldsymbol{x}$ satisfies equation (2.9) the equation of the pencil of conics (2.8) reads

$$y_1^2 + (F_{(1)}F - 1)y_1y_2 + (F_{(2)}F - 1)y_1y_3 + \lambda y_2y_3 = 0. \tag{4.12}$$

When the parameter λ equals

$$\lambda_0 = \frac{F}{F_{(12)}} + 1 - F_{(1)}F - F_{(2)}F, \tag{4.13}$$

the conic passes through $\boldsymbol{x}_{1(11)}$ and $\boldsymbol{x}_{-1(22)}$.

Instead of the basis $\boldsymbol{x}$, $\boldsymbol{x}_{-1(2)}$ and $\boldsymbol{x}_{1(1)}$ of the plane of the elementary quadrilateral of $\boldsymbol{x}$ let us choose points $\boldsymbol{x}$, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, where

$$\boldsymbol{u}_1 = \Delta_1\left(\frac{\boldsymbol{x}}{\phi}\right), \qquad \boldsymbol{u}_2 = \Delta_2\left(\frac{\boldsymbol{x}}{\phi}\right), \tag{4.14}$$

represent points of intersection of the tangent lines of $\boldsymbol{x}$ and $\boldsymbol{x}'$. Moreover the line $L_{\boldsymbol{u}_1\boldsymbol{u}_2}$ is the intersection of the planes of elementary quadrilaterals of $\boldsymbol{x}$ and $\boldsymbol{x}'$. The transition formulas read

$$\boldsymbol{x}_{-1(2)} = F_{(1)}\phi_{(1)}\boldsymbol{u}_1 + \boldsymbol{x}\left(\frac{F_{(1)}\phi_{(1)}}{\phi} - 1\right), \tag{4.15}$$

$$\boldsymbol{x}_{1(1)} = F_{(2)}\phi_{(2)}\boldsymbol{u}_2 + \boldsymbol{x}\left(\frac{F_{(2)}\phi_{(2)}}{\phi} - 1\right). \tag{4.16}$$

When (t_1, t_2, t_3) are coordinates of a point

$$\boldsymbol{y} = t_1\boldsymbol{x} + t_2\boldsymbol{u}_1 + t_3\boldsymbol{u}_2, \tag{4.17}$$

then the equation of the pencil (4.13) is transformed into

$$t_1^2 + a_1b_1t_2^2 + a_2b_2t_3^2 + t_1t_2(a_1 + b_1) + t_1t_3(a_2 + b_2) + t_2t_3(a_1b_2 + a_2b_1 + \mu) = 0, \tag{4.18}$$

where

$$a_i = \frac{1}{F_{(i)}\phi_{(i)}} - \frac{1}{\phi}, \qquad b_i = \frac{F}{\phi_{(i)}} - \frac{1}{\phi}, \quad i = 1, 2, \tag{4.19}$$

and

$$\mu = \frac{\lambda}{F_{(1)}F_{(2)}\phi_{(1)}\phi_{(2)}}. \tag{4.20}$$

The involution on the line $t_1 = 0$ is equivalent to the problem of finding the second root of the quadratic equation

$$a_1b_1t_2^2 + a_2b_2t_3^2 + t_2t_3(a_1b_2 + a_2b_1 + \mu) = 0, \tag{4.21}$$

when the first root is given. The double points of the involution correspond to

$$\mu = -(a_1b_2 + a_2b_1) \pm 2\sqrt{a_1b_1a_2b_2}, \tag{4.22}$$

and their coordinates are solutions of the equation

$$(\sqrt{a_1b_1}t_2 \pm \sqrt{a_2b_2}t_3)^2 = 0. \tag{4.23}$$

Finally, the double points are given by

$$\sqrt{a_1b_1}\boldsymbol{u}_2 \pm \sqrt{a_2b_2}\boldsymbol{u}_1. \tag{4.24}$$

To find the equation of the second pencil of conics we take the primed version of equation (4.18) with

$$\boldsymbol{u}_1' = \Delta_1\left(\frac{\boldsymbol{x}'}{\phi'}\right) = \theta_{(1)}\theta_{(12)}\boldsymbol{u}_1, \tag{4.25}$$

$$\boldsymbol{u}_2' = -\Delta_2\left(\frac{\boldsymbol{x}'}{\phi'}\right) = \theta_{(2)}\theta_{(12)}\boldsymbol{u}_2, \tag{4.26}$$

and

$$a_i' = \frac{1}{F_{(i)}'\phi_{(i)}'} - \frac{1}{\phi'} = -\theta_i^2 a_i, \qquad b_i' = \frac{F'}{\phi_{(i)}'} - \frac{1}{\phi'} = -\frac{\theta_1\theta_2\theta_{12}}{\theta} b_i \quad i = 1,2. \tag{4.27}$$

The double points on the line $t_1' = 0$ of the second involution are given by

$$\sqrt{a_1'b_1'}\boldsymbol{u}_2' \pm \sqrt{a_2'b_2'}\boldsymbol{u}_1' = \frac{(\theta_{(1)}\theta_{(2)}\theta_{(12)})^{3/2}}{(\theta)^{1/2}}(\sqrt{a_1b_1}\boldsymbol{u}_2 \pm \sqrt{a_2b_2}\boldsymbol{u}_1), \tag{4.28}$$

and due to Theorem 4.2, both evolutions are the same. □

Proposition 4.8. *Given Koenigs lattice $\boldsymbol{x}$ and its transform $\boldsymbol{x}'$ then any pair of intersecting conics of two pencils determines a pencil of quadrics. Such a pencil defines on the line $L_{\boldsymbol{x}\boldsymbol{x}'}$ of the congruence the involution described in Proposition 4.3.*

Proof. Let us choose points $\boldsymbol{x}$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{x}'$ as the vertices of the reference frame, then (t_1, t_2, t_3, t_4) are coordinates of a point

$$\boldsymbol{y} = t_1\boldsymbol{x} + t_2\boldsymbol{u}_1 + t_3\boldsymbol{u}_2 + t_4\boldsymbol{x}'. \tag{4.29}$$

Notice the following relation between these coordinates and the coordinates t_i' used in the last part of the proof above

$$t_1' = t_4, \qquad t_2' = \frac{t_2}{\theta_{(1)}\theta_{(12)}}, \qquad t_3' = -\frac{t_3}{\theta_{(2)}\theta_{(12)}}. \tag{4.30}$$

The transformation formulas (4.30) imply that the equation of the second pencil reads

$$\begin{split} \tfrac{\theta_{(1)}\theta_{(2)}}{\theta\theta_{(12)}}\left[a_1b_1t_2^2 + a_2b_2t_3^2 - t_2t_3\left(\tfrac{\theta_{(1)}}{\theta_{(2)}}a_1b_2 + \tfrac{\theta_{(2)}}{\theta_{(1)}}a_2b_1 + \tfrac{\mu'\theta}{\theta_{(1)}^2\theta_{(2)}^2\theta_{(12)}}\right)\right] + \\ -t_2t_4\left(\tfrac{\theta_{(1)}}{\theta_{(12)}}a_1 + \tfrac{\theta_{(2)}}{\theta}b_1\right) + t_3t_4\left(\tfrac{\theta_{(2)}}{\theta_{(12)}}a_2 + \tfrac{\theta_{(1)}}{\theta}b_2\right) + t_4^2 = 0, \end{split} \tag{4.31}$$

and, therefore, the relation between parameters of the corresponding conics in the pencils reads

$$\frac{\phi}{\theta_{(2)}}a_1b_2 + \frac{\phi}{\theta_{(1)}}a_2b_1 + \frac{\mu'\theta}{\theta_{(1)}^2\theta_{(2)}^2\theta_{(12)}} + \mu = 0. \tag{4.32}$$

Any two corresponding conics of the pencils (4.18) and (4.31) define a pencil of quadrics and, finally, we obtain the following two-parameter linear system of quadrics

$$\begin{split} t_1^2 + a_1b_1t_2^2 + a_2b_2t_3^2 + t_1t_2(a_1 + b_1) + t_1t_3(a_2 + b_2) + t_2t_3(a_1b_2 + a_2b_1 + \mu) + \\ + \tfrac{\theta\theta_{(12)}}{\theta_{(1)}\theta_{(2)}}t_4^2 + \nu t_1t_4 - t_2t_4\left(\tfrac{\theta}{\theta_{(2)}}a_1 + \tfrac{\theta_{(12)}}{\theta_{(1)}}b_1\right) + t_3t_4\left(\tfrac{\theta}{\theta_{(1)}}a_2 + \tfrac{\theta_{(12)}}{\theta_{(2)}}b_2\right) = 0. \end{split} \tag{4.33}$$

Notice that any pair of two conics of the pencils (i.e., an arbitrary fixed μ in equation (4.33)) determines the same involution on the line $t_2 = t_3 = 0$ (the line of the congruence)

$$t_1^2 + \nu t_1t_4 + \frac{\theta\theta_{(12)}}{\theta_{(1)}\theta_{(2)}}t_4^2 = 0. \tag{4.34}$$

The fixed points of the involution correspond to

$$\nu = \pm\sqrt{\frac{\theta\theta_{(12)}}{\theta_{(1)}\theta_{(2)}}}, \tag{4.35}$$

and read

$$\pm\sqrt{\theta\theta_{(12)}}\boldsymbol{x} + \sqrt{\theta_{(1)}\theta_{(2)}}\boldsymbol{x}'. \tag{4.36}$$

One can check using equations (4.36) and (4.1) that the pairs $\boldsymbol{x}$ and $\boldsymbol{x}'$, $\boldsymbol{y}_1$ and $\boldsymbol{y}_{1(1)}$, and $\boldsymbol{y}_2$ and $\boldsymbol{y}_{2(2)}$ are harmonically conjugate with respect to the fixed points. □

Corollary 4.9. *Equations* (4.13), (4.20)*, their primed versions and equation* (4.32) *imply that the distinguished six-point conics of two pencils intersect.*

5 The permutability of superpositions of the discrete Koenigs transformations

We show in this section that superpositions of the discrete Koenigs transformations satisfy the permutability property. We start with the relevant properties of the Moutard equation and then we present the permutability theorem for the discrete Koenigs transformation proving this way the integrability of the Koenigs lattice.

5.1 The discrete Moutard transformation and its permutability property

We recall the known material on the Darboux type transformation for the Moutard equation [30, 33] and the permutability theorem for this transformation [8, 28]. The novelty here is the presentation of the Moutard transformation within the general setting of the fundamental transformation of quadrilateral lattices.

Consider the Moutard lattice, i.e., the quadrilateral lattice whose homogeneous coordinates $\boldsymbol{y} : \mathbb{Z}^2 \to \mathbb{R}^{N+1}_*$ satisfy (up to a gauge) the discrete Moutard equation (3.10)

$$\boldsymbol{y}_{(12)} + \boldsymbol{y} = F(\boldsymbol{y}_{(1)} + \boldsymbol{y}_{(2)}). \tag{5.1}$$

Let θ be a scalar solution of this equation, linearly independent of the components of $\boldsymbol{y}$. One can check that the function $\psi = \theta_{(-1)} + \theta_{(-2)}$ satisfies the equation

$$\psi_{(12)} + \psi = F_{(-2)}\psi_{(1)} + F_{(-1)}\psi_{(2)}, \tag{5.2}$$

adjoint to the Moutard equation. Notice that, like in the Koenigs lattice case, to construct the fundamental transformation of the Moutard lattice we need only half of the data, but this time the solution of the Moutard equation gives a solution of its adjoint. The solution of the system (3.4) (with the change of notation $\phi \to \theta$ and $\theta \to \psi$) reads then

$$k = \theta\theta_{(-1)}, \qquad \ell = -\theta\theta_{(-2)}, \tag{5.3}$$

and the lattice $\hat{\boldsymbol{y}}$, the solution of the linear system

$$\Delta_1(\hat{\boldsymbol{y}}) = \theta\theta_{(1)}\Delta_1\left(\frac{\boldsymbol{y}}{\theta}\right), \tag{5.4}$$

$$\Delta_2(\hat{\boldsymbol{y}}) = -\theta\theta_{(2)}\Delta_2\left(\frac{\boldsymbol{y}}{\theta}\right), \tag{5.5}$$

satisfies the Laplace equation

$$\hat{\boldsymbol{y}}_{(12)} = F\frac{\theta_{(2)}}{\theta}\hat{\boldsymbol{y}}_{(1)} + F\frac{\theta_{(1)}}{\theta}\hat{\boldsymbol{y}}_{(2)} + \left(1 - F\frac{\theta_{(2)}}{\theta} - F\frac{\theta_{(1)}}{\theta}\right)\hat{\boldsymbol{y}}. \tag{5.6}$$

Its gauge transform $\boldsymbol{y}'$ defined by

$$\hat{\boldsymbol{y}} = \boldsymbol{y}'\theta \tag{5.7}$$

satisfies the Moutard equation with the potential F' given by equation (3.14).

Finally, we obtain the known [33] formulas

$$\Delta_1(\theta\boldsymbol{y}') = \theta\theta_{(1)}\Delta_1\left(\frac{\boldsymbol{y}}{\theta}\right), \tag{5.8}$$

$$\Delta_2(\theta\boldsymbol{y}') = -\theta\theta_{(2)}\Delta_2\left(\frac{\boldsymbol{y}}{\theta}\right), \tag{5.9}$$

which allow to find the new Moutard lattice $\boldsymbol{y}'$ given the old Moutard lattice $\boldsymbol{y}$ and the scalar solution θ of the Moutard equation of $\boldsymbol{y}$. Notice that $\theta' = 1/\theta$ satisfies the Moutard equation of $\boldsymbol{y}'$.

Let θ^1 and θ^2 be two solutions of the Moutard equation (3.10). Denote by $\boldsymbol{y}^{(1)}$ and $\theta^{2(1)}$ the transforms of $\boldsymbol{y}$ and θ^2 via θ^1 and denote by $\boldsymbol{y}^{(2)}$ and $\theta^{1(2)}$ the transforms of $\boldsymbol{y}$ and θ^1 via θ^2. Then $\boldsymbol{y}^{(1)}$, $\theta^{2(1)}$ and $\theta^{1(1)} = 1/\theta^1$ satisfy the Moutard equation with the potential

$$F^{(1)} = F\frac{\theta^1_{(1)}\theta^1_{(2)}}{\theta^1\theta^1_{(12)}}, \tag{5.10}$$

and $\boldsymbol{y}^{(2)}$, $\theta^{1(2)}$ and $\theta^{2(2)} = 1/\theta^2$ satisfy the Moutard equation with the potential

$$F^{(2)} = F\frac{\theta^2_{(1)}\theta^2_{(2)}}{\theta^2\theta^2_{(12)}}. \tag{5.11}$$

Notice [8, 28] that the transformation formulas (5.8) give

$$\Delta_1(\theta^1\theta^{2(1)}) = -\Delta_1(\theta^2\theta^{1(2)}), \tag{5.12}$$

$$\Delta_2(\theta^1\theta^{2(1)}) = -\Delta_2(\theta^2\theta^{1(2)}), \tag{5.13}$$

which implies that fixing one of the two integration constants we have

$$\theta^1\theta^{2(1)} = -\theta^2\theta^{1(2)} = \Xi. \tag{5.14}$$

Then the lattices $\boldsymbol{y}^{(12)}$ of the one parameter family (due to the an additive constant in Ξ) given by

$$\boldsymbol{y}^{(12)} = \boldsymbol{y} + \frac{\theta^1\theta^2}{\Xi}(\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(2)}), \tag{5.15}$$

are simultaneously transforms of $\boldsymbol{y}^{(1)}$ via $\theta^{2(1)}$ and transforms of $\boldsymbol{y}^{(2)}$ via $\theta^{1(2)}$.

Remark. To obtain symmetric more form of the superposition formula (5.15) one can use the allowed gauge freedom in the transformation formulas [8, 28].

5.2 Superposition of the discrete Koenigs transformations

Let us use θ^1 and θ^2 to find two transforms of the Koenigs lattice $\boldsymbol{x}$ satisfying equation (2.9). According to notation of Proposition 3.6 denote by $\phi^1 = \theta^1_{(1)} + \theta^1_{(2)}$ and $\phi^2 = \theta^2_{(1)} + \theta^2_{(2)}$ the corresponding solutions of the Koenigs lattice equation. Denote by $\boldsymbol{x}^{(1)}$ and $\phi^{2(1)}$ the transforms of $\boldsymbol{x}$ and ϕ^2 with respect to θ^1, i.e.,

$$\Delta_1\left(\frac{1}{\phi^{1(1)}}\begin{pmatrix}\boldsymbol{x}^{(1)}\\ \phi^{2(1)}\end{pmatrix}\right) = (\theta^1\theta^1_{(2)})_{(1)}\Delta_1\left(\frac{1}{\phi^1}\begin{pmatrix}\boldsymbol{x}\\ \phi^2\end{pmatrix}\right), \tag{5.16}$$

$$\Delta_2\left(\frac{1}{\phi^{1(1)}}\begin{pmatrix}\boldsymbol{x}^{(1)}\\ \phi^{2(1)}\end{pmatrix}\right) = -(\theta^1\theta^1_{(1)})_{(2)}\Delta_2\left(\frac{1}{\phi^1}\begin{pmatrix}\boldsymbol{x}\\ \phi^2\end{pmatrix}\right), \tag{5.17}$$

where

$$\phi^{1(1)} = \frac{1}{\theta^1_{(1)}} + \frac{1}{\theta^1_{(2)}}. \tag{5.18}$$

According to Proposition 3.6 the functions $\boldsymbol{x}^{(1)}$, $\phi^{2(1)}$ and $\phi^{1(1)}$ satisfy the Koenigs lattice equation with the transformed potential $F^{(1)}$ given by (5.10). Similarly, by $\boldsymbol{x}^{(2)}$ and $\phi^{1(2)}$ denote the transforms of $\boldsymbol{x}$ and ϕ^1 with respect to θ^2, i.e.,

$$\Delta_1\left(\frac{1}{\phi^{2(2)}}\begin{pmatrix}\boldsymbol{x}^{(2)}\\ \phi^{1(2)}\end{pmatrix}\right) = (\theta^2\theta^2_{(2)})_{(1)}\Delta_1\left(\frac{1}{\phi^2}\begin{pmatrix}\boldsymbol{x}\\ \phi^1\end{pmatrix}\right), \tag{5.19}$$

$$\Delta_2\left(\frac{1}{\phi^{2(2)}}\begin{pmatrix}\boldsymbol{x}^{(2)}\\ \phi^{1(2)}\end{pmatrix}\right) = -(\theta^2\theta^2_{(1)})_{(2)}\Delta_2\left(\frac{1}{\phi^2}\begin{pmatrix}\boldsymbol{x}\\ \phi^1\end{pmatrix}\right), \tag{5.20}$$

where

$$\phi^{2(2)} = \frac{1}{\theta^2_{(1)}} + \frac{1}{\theta^2_{(2)}}, \tag{5.21}$$

and $\boldsymbol{x}^{(2)}$, $\phi^{1(2)}$ and $\phi^{2(2)}$ satisfy the Koenigs lattice equation with the transformed potential given by (5.11).

Proposition 5.1. *The lattices $\boldsymbol{x}^{(12)}$ of the one parameter family (because of the free parameter in the definition of Ξ) given by*

$$\boldsymbol{x}^{(12)} = -\frac{\Xi_{(1)}\Xi_{(2)}\phi^{1(21)}\phi^{2(12)}}{\phi^1\phi^2}\boldsymbol{x} + \frac{\phi^{1(21)}}{\phi^{1(1)}}\boldsymbol{x}^{(1)} + \frac{\phi^{2(12)}}{\phi^{2(2)}}\boldsymbol{x}^{(2)}, \tag{5.22}$$

where

$$\phi^{1(21)} = \frac{1}{\theta^{1(2)}_{(1)}} + \frac{1}{\theta^{1(2)}_{(2)}}, \qquad \phi^{2(12)} = \frac{1}{\theta^{2(1)}_{(1)}} + \frac{1}{\theta^{2(1)}_{(2)}}, \tag{5.23}$$

are simultaneously the Koenigs transforms of $\boldsymbol{x}^{(1)}$ via $\theta^{2(1)}$ and the Koenigs transforms of $\boldsymbol{x}^{(2)}$ via $\theta^{1(2)}$.

Proof. We have to check that $\boldsymbol{x}^{(12)} = \boldsymbol{x}^{(21)}$ defined in (5.22) satisfies equations

$$\Delta_1\left(\frac{\boldsymbol{x}^{(12)}}{\phi^{2(12)}}\right) = (\theta^{2(1)}\theta^{2(1)}_{(2)})_{(1)}\Delta_1\left(\frac{\boldsymbol{x}^{(1)}}{\phi^{2(1)}}\right), \tag{5.24}$$

$$\Delta_2\left(\frac{\boldsymbol{x}^{(12)}}{\phi^{2(12)}}\right) = -(\theta^{2(1)}\theta^{2(1)}_{(1)})_{(2)}\Delta_2\left(\frac{\boldsymbol{x}^{(1)}}{\phi^{2(1)}}\right), \tag{5.25}$$

which define the transform $\boldsymbol{x}^{(12)} = (\boldsymbol{x}^{(1)})^{(2)}$ of $\boldsymbol{x}^{(1)}$ via $\theta^{2(1)}$, and satisfies equations

$$\Delta_1\left(\frac{\boldsymbol{x}^{(21)}}{\phi^{1(21)}}\right) = (\theta^{1(2)}\theta^{1(2)}_{(2)})_{(1)}\Delta_1\left(\frac{\boldsymbol{x}^{(2)}}{\phi^{1(2)}}\right), \tag{5.26}$$

$$\Delta_2\left(\frac{\boldsymbol{x}^{(21)}}{\phi^{1(21)}}\right) = -(\theta^{1(2)}\theta^{2(2)}_{(1)})_{(2)}\Delta_2\left(\frac{\boldsymbol{x}^{(2)}}{\phi^{1(2)}}\right), \tag{5.27}$$

which define the transform $\boldsymbol{x}^{(21)} = (\boldsymbol{x}^{(2)})^{(1)}$ of $\boldsymbol{x}^{(2)}$ via $\theta^{1(2)}$. This can be done by direct verification using equations (5.12)–(5.14) and (5.16)–(5.21). □

Remark. To obtain the superposition formula (5.22) we assume that $\boldsymbol{x}^{(21)} = \boldsymbol{x}^{(12)}$ and formulas (5.24) – (5.27) hold.

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